Math 3235 Probability Theory

9/1/22

Conditional Probability

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$$

$\mathbb{P}(\cdot\,|B)$ is a probability function

Independent Events

$$\mathbb{P}(A\cap B) = \mathbb{P}(A)\,\mathbb{P}(B) \qquad A \perp\!\!\!\perp B$$

$$\Downarrow$$

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

Fam A_i of events, $i \in I$

A_i are independent if

$$\mathbb{P}\Big(\bigcap_{j\in J} A_j\Big) = \prod_{j\in J} \mathbb{P}(A_j) \qquad \forall J \subset I$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$$

$\mathbb{P}(A)$ $\qquad$ $\mathbb{P}(B \mid A)$ $\qquad$ $\mathbb{P}(B \mid A^c)$

$\mathbb{P}(B^c \mid A)$

$\mathbb{P}(A \mid B) =$ Sick when Test P

$\mathbb{P}(B \mid A) =$ P if Sick

$\mathbb{P}(B \mid A^c) =$ P if Healthy

$$\mathbb{P}(B) = \mathbb{P}(B \mid A)\,\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\,\mathbb{P}(A^c)$$

Th: A_i disjoint exhaustive $\quad \bigcup_i A_i = \Omega$

Partition of Ω.

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B \mid A_i)\,\mathbb{P}(A_i)$$

$$\sum_i \mathbb{P}(B \mid A_i)\mathbb{P}(A_i) =$$

$$\sum_i \mathbb{P}(B \cap A_i) = \boxed{(A_i \cap B) \cap (A_j \cap B) = \phi}$$

$$\mathbb{P}\left(\bigcup_i (B \cap A_i)\right) = \mathbb{P}(B)$$

$$\bigcup_i (B \cap A_i) = B \cap \bigcup_i A_i = B \cap \Omega = B$$

Bayes Theorem:

A_i partition of Ω

B

$$\mathbb{P}(A_i \mid B) = \frac{\mathbb{P}(B \mid A_i)\,\mathbb{P}(A_i)}{\sum_j \mathbb{P}(B \mid A_j)\mathbb{P}(A_j)}$$

The family A_i countable.

Urn with 10 balls.

Some of the balls are blue

The other are red.

$n \in \{0, \dots, 10\}$ blue

Pryor: $p(n) = \frac{1}{11} \quad \forall n$

Pick a ball at random and

show it to you.

A_n = n blue ball in the urn

Partition

$\mathbb{P}(A_n)$ I want $\mathbb{P}(A_n \mid B)$

B result of the extraction.

$$\mathbb{P}(A_n \mid B) = \frac{\mathbb{P}(B \mid A_n)\,\mathbb{P}(A_n)}{\mathbb{P}(B)}$$

$$\mathbb{P}(B) = \sum_{m=0}^{10} \mathbb{P}(B \mid A_m)\,\mathbb{P}(A_m)$$

$\mathbb{P}(B \mid A_n) = \frac{n}{10}$ $\mathbb{P}(A_n)$ prior

$$\frac{\qquad\qquad}{\qquad} \, \partial \, \frac{\qquad\qquad}{\qquad}$$

Probability Space.

Random Variable

Rolling 1 dice.

$\Omega = \{ (i, j) \mid i, j = 1, \dots 6 \}$

$i + j \; : \; \Omega \to \mathbb{R}$

if $i+j \geq 7$ $\quad 1$

if $i+j < 7$ $\quad 0$

$\Omega \to \mathbb{R}$

Coin flip

Ω = all sequences of H and T

$\sigma = (H H T H T T \dots)$

Given a sequence σ, the position of the first H $: \Omega \to \mathbb{R}$

$\sigma = (\sigma_1 \; \sigma_2 \; \dots \; \sigma_n \; \dots)$ $\quad H = 0$

$T = 1$

$$X(\sigma) = \sum_{n=1}^{\infty} \sigma_n 2^{-n}$$

Random Variable is a function from Ω into $\mathbb{R}$.

Notation: X. Y Z

N T

$\{\omega \in \Omega \ , \ X(\omega) = x\} = A_x$

X is a discrete r.v. if $\mathrm{Im}(X)$ is contable.

$0.011 \ldots 1 \ldots = 0.1$

$01 \ldots 1 \ldots$

$10 \ldots 0$

Im (X) contable set

$x \in \mathrm{Im}(X)$

$p(x) = \mathbb{P}(X = x) = \mathbb{P}(\{\omega \mid X(\omega) = x\})$

$A_x = \{\omega \mid X(\omega) = x\}$

$A_x \in \mathcal{F}$

The counter image of any

$x \in \mathrm{Im}(X)$ is an event.

A discrete r.v. X is a function

function $\Omega \to \mathbb{R}$ such that

$\mathrm{Im}(X)$ is countable

$X^{-1}(x) \in \mathcal{F} \quad \forall\ x \in \mathrm{Im}(X)$

$(X^{-1}(y) = \phi$ if $y \notin \mathrm{Im}(X)$

$\phi \in \mathcal{F}$)

$\Gamma = \mathrm{Im}(X)$ contable

$\mathcal{G}$ = power set of Γ

$X^{-1}(A) \in \mathcal{F} \quad \forall A \in \mathcal{G}$